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Report on the doctoral dissertation of Marta Piecyk, M.Sc.,
"Graph Homomorphism – Exploring the Boundaries of Tractability"

The dissertation is written under supervision of Professors Zbigniew Lonc, Ph.D., D.Sc., and Paweł Rzążewski, Ph.D., D.Sc., in English language. The dissertation is in the domain of graph theory, addressing mainly the graph homomorphism problem, a generalization of the well-studied k-COLORING problem, denoted by HOM(H) where H is a fixed target graph. Whenever H is non-bipartite graph without loops, the problem is known to be NP-hard. A generalization of the homomorphism problem, called the list homomorphism problem, is also considered; here, every vertex of the input graph is equipped with a list of vertices of H to which it can be mapped.

Structure of the doctoral dissertation

The doctoral dissertation is well organized and well structured. The introduction of the dissertation gives the motivation for the study, defines the main concepts (graph coloring, cutwidth, diameter, graph homomorphisms), and explains the main results of the dissertation. Roughly speaking, these are divided into two parts, depending on which parameter of the input graph governs the complexity of the homomorphism problem and its list variant: the cutwidth or the diameter.

Chapter 2 collects the necessary preliminaries, introducing the basic notions and formally defining the computational decision problems of interest.

In Chapter 3 the necessary technical background is presented, consisting of some of the existing results on the homomorphism and the list homomorhpism problems.

Chapter 4 is based on an ICALP 2024 paper published by the candidate in coauthorship with Carla Groenland, Isja Mannens, Jesper Nederlof, and Paweł Rzążewski, and considers the HOM(H) problem parameterized by the cutwidth of the input graph.

In Chapter 5, the author focuses on graphs with bounded diameter, and first considers the 3-Coloring problem and then the general Hom(H) problem. The chapter is based on two papers of the candidate, one published in SIAM Journal on Discrete Mathematics in 2022 and coauthored with Michał Dębski and Paweł Rzążewski, and one published as a single author in the proceedings of MFCS 2024.

In Chapter 6 some other interesting results coauthored by the candidate are presented, which were not selected for inclusion in the dissertation. These results are on a number of topics in graph theory, including list homomorphisms and treewidth, precoloring extension and forbidden induced subgraphs, and graph reconstruction.

In the dissertation, some results from other papers of the author are also used, which were submitted before the candidate started their doctoral studies. These are an ESA 2020 paper that the candidate coauthored with Karolina Okrasa and Paweł Rzążewski, and a STACS 2021 paper coauthored with Paweł Rzążewski, which is based on the Master thesis of the author.

As is common for contemporary research in mathematics, parts of the dissertation are based on papers that were obtained in collaboration with other coauthors. Given that the second part of Chapter 5 is based on a single-author paper and the overall quality and coherence of the presentation of the dissertation, I have no doubts about the high level of the individual contribution of the candidate to the work.

Literature used

The dissertation features an extensive literature of over 130 references, covering both references essential for further background as well as specialized research papers on the topics studied, giving thus an actual, complete, and relevant account of bibliography in the area of study, and indicating the candidate's broad overview of the topic of the dissertation and related areas.

Literature used includes many recent works from top journals and conferences in discrete mathematics and theoretical computer science, some references from linear algebra, as well as a number of older, classical papers, such as Fekete's lemma, a paper by Erdős and Szekeres on upper bounds on multi-colored Ramsey numbers, Shannon's paper on the zero error capacity of a noisy channel, Mycielski's contruction on triangle-free graphs with large chromatic number, Lovász's paper on the Shannon capacity of graphs, Korshunov's paper showing that almost all graphs have diameter 2, and Edwards' paper reducing List k-Coloring with all lists of size at most two to 2-SAT.

The literature cited in the introductory chapter is related to an overview of recent approaches to understanding the boundaries between tractable and hard problems, including parameterized complexity, as well as restrictions on input instances, such as forbidding some fixed graph or some family of graphs as induced subgraphs, intersection graphs of geometric objects, or bounding some parameter of the input graph. These include treewidth, pathwidth, treedepth, cliquewidth, cutwidth, twinwidth, shrubdepth, and branchdepth, distances to some fixed hereditary graph classes, or some variations of well-known parameters, such as $\mathcal{H}$-elimination distance, $\mathcal{H}$-treewidth, or tree-independence number. Other literature cited in the introduction is related to the Exponential Time Hypothesis (ETH) and the Strong Exponential Time Hypothesis (SETH), to graph coloring and graph homomorphism problems, matrix multiplication, bounded diameter graphs, and asymptotic induced matching number of hypergraphs.

In the main part of the dissertation, literature used includes literature on representative sets, product dimension (also known as Prague dimension), biclique covering number and related parameters, projective graphs, Shannon capacity, graph homomorphism and list homomorphism problems, direct product of graphs, half-induced matchings, minrank of graphs and support-rank of matrices, 2-SAT, Chernoff bounds, and on parameterized and fine-grained complexity. The literature also includes three books on algorithms and complexity: *Introduction to Algorithms* by Cormen, Leiserson, Rivest and Stein, *Computational Complexity* by Papadimitriou, and *Parameterized Algorithms* by Cygan, Fomin, Kowalik, Lokshtanov, Marx, Pilipczuk, Pilipczuk, and Saurabh.

The aim of the dissertation

For a non-bipartite simple graph H, the graph homomorphism, $\textsc{Hom}(H)$, problem is known to be NP-complete. The main aim of the dissertation is to identify restrictions of the input instances of the problem under which the problem can be solved in polynomial or FPT time, with a focus on two parameters: the cutwidth and the diameter. The cutwidth of a graph G, denoted by $\mathrm{ctw}(G)$, is defined as the minimum, over all linear orderings of the vertex set of G, of the maximum number of edges leaving a prefix of the ordering. The diameter of G is the maximum distance between any two vertices in G. It turns out that the $\textsc{Hom}(H)$ problem exhibits very different behavior with respect to these two parameters.

Cutwidth. Jansen and Nederlof [TCS, 2019] proved that every instance G of k-$\textsc{Coloring}$ can be solved in (randomized) time $2^{\mathrm{ctw}(G)}n^{\mathcal{O}(1)}$. The aim of this part of the dissertation is to find, for every graph H, a constant c_H such that every instance G of $\textsc{Hom}(H)$ can be solved in time $c_H^{\mathrm{ctw}(G)}n^{\mathcal{O}(1)}$, but there is no algorithm running in time $(c_H - \epsilon)^{\mathrm{ctw}(G)}n^{\mathcal{O}(1)}$, for any $\epsilon > 0$, unless the SETH fails. While this problem is not solved entirely, an asymptotic parameter is defined, called $\mathrm{mimsup}(H)$, that seems to be a good candidate for c_H. The parameter $\mathrm{mimsup}(H)$ is defined in terms of the sizes of maximum induced matchings in powers of H with respect to the direct product. Via the square of the line graph operator, the parameter is related to the Shannon capacity of graphs. For a closely related parameter $\mathrm{mimsup}^*(H)$, it is shown in the dissertation that the maximum number of states in a natural dynamic programming is precisely $\mathrm{mimsup}^*(H)^{\mathrm{ctw}(G)}$. On the other hand, it is proved that for almost every graph H (and one can remove "almost" assuming two conjectures from early 2000's), there is no algorithm solving every instance G of $\textsc{Hom}(H)$ in time $(\mathrm{mimsup}^*(H) - \epsilon)^{\mathrm{ctw}(G)}n^{\mathcal{O}(1)}$, for any $\epsilon > 0$, unless the SETH fails. An algorithm is also given running in time $e^{\mathcal{O}(\mathrm{mimsup}^*(H)\cdot \mathrm{ctw}(G)\cdot \log \mathrm{ctw}(G))} \cdot n^{\mathcal{O}(1)}$.

Diameter. For diameter-2 graphs, it is known that for any $k \geq 4$, k-$\textsc{Coloring}$ cannot be solved in subexponential time, unless the ETH fails, while for $k = 3$, a subexponential-time algorithm running in time $2^{\mathcal{O}(\sqrt{n \log n})}$ is known, but the existence of a polynomial-time algorithm is still an open question. For diameter-3 graphs, 3-$\textsc{Coloring}$ is known to be NP-complete. The aim of this part of the dissertation is to show that $\textsc{List}$ 3-$\textsc{Coloring}$ can be solved in time $2^{\mathcal{O}(n^{2/3} \log^{2/3} n)}$ on n-vertex diameter-3 graphs and in time $2^{\mathcal{O}(n^{1/3} \log^2 n)}$ on n-vertex diameter-2 graphs. In fact, the diameter-2 result is generalized to the weighted version of the problem, while for the diameter-3 case, an ETH-based lower bound for the weighted case is derived. Furthermore, for the more general $\textsc{Hom}(H)$ problem, it is shown that if H is triangle-free, then $\textsc{Hom}(H)$ can be solved in polynomial time on diameter-2 graphs. Finally, for the case when H is C_{2k+1}, the cycle on $2k+1$ vertices, and $k \geq 2$, it is proved that every diameter-$(k+1)$ instance of $\textsc{Hom}(C_{2k+1})$ can be solved in polynomial time, every n-vertex diameter-$(k+2)$ instance of $\textsc{Hom}(C_{2k+1})$ be solved in time $e^{\mathcal{O}((n \log n)^{(k+1)/(k+2)})}$, and there is no algorithm that solves every diameter-$(2k+2)$ instance of $\textsc{Hom}(C_{2k+1})$ in subexponential time, unless the ETH fails.

The aims of the dissertation are worthy and provide insightful new results regarding the boundaries of tractability for the graph homomorphism problem. Moreover, the results related to cutwidth represent the first time when the parameterized complexity of a problem is linked to an asymptotic parameter.

Used research methods

As indicated by the wide variety of the literature used, the dissertation uses methods, concepts, and tools from a number of areas of mathematics and computer science. While a precise assignment of methods to areas is difficult to make, an approximate (and probably incomplete) list can be summarized as follows:

- *graph theory*, including notions of graph parameters (most notably cutwidth and diameter), graph homomorphisms, graph cores, and graph products (direct product, strong product), with connections to Prague dimension and Shannon capacity of graphs,
- *probability*, via randomized constructions to obtain matrices or sets of vertices in a graph satisfying certain properties; the constructions are carefully analyzed, in one case also using Chernoff concentration bounds,
- *linear algebra*, including basic notions such as rank and Gaussian elimination, as well as matrix products such as the Kronecker product and the Hadamard product,
- *theoretical computer science*, including tools from algorithm theory, such as dynamic programming, and branching algorithms, as well as from computational complexity, both classical (reductions for proving NP-hardness) as well as fine-grained (reductions for proving (S)ETH-based lower bounds), and
- *combinatorics*, for example, by addressing the Binary Constraint Satisfaction Problems and analyzing the running times of recursive algorithms using recursive inequalities and multinomial coefficients.

Of course, classical proof techniques in discrete mathematics such as proof by induction and proof by contradiction also play an important role in the dissertation.

When reading the dissertation, I appreciated not only the large variety of tools and techniques used, but also the ease with which tools from one area were applied in another. To mention just one such example, in Section 4.2.3, the support rank of the adjacency matrix of a non-bipartite graph H is bounded by constructing particular biclique covers of the bipartite complement of a bipartite graph associated to H. I was also impressed by the skillful use of probability arguments and calculations, branching algorithms, and clever constructions used in the hardness proofs.

Discussion of research results in the doctoral dissertation

The dissertation is very carefully written. Arguments are spelled out in detail and proofs are clearly structured and presented, often equipped with helpful intuitive explanations. The key concepts and some of the proofs are illustrated with helpful figures. The obtained research results are clearly placed in the context of existing work. Remarkably, except for a couple of typos, I did not find any mistakes, incorrect statements, or imprecise statements. I have no questions or remarks regarding the content or clarity of the presentation.

Information on practical applications of the obtained research results

The research carried out in the dissertation is basic and theoretical in nature, hence, practical applications are not discussed. Nevertheless, it is worth mentioning that the graph homomorphism problem generalizes the classical k-Coloring problem, which is one of the most well studied graph problems, with many practical applications, such as frequency assignment in radio networks or compiler optimization, to mention just two.

Overall evaluation

This is an excellent dissertation based on a technically impressive research work, presenting original and clearly presented solutions to a number of research problems. The dissertation is rich in results and provides valuable new insights regarding the boundaries of tractability for variants of the graph homomorphism problem. This includes both efficient (polynomial, fixed-parameter tractable, or subexponential) algorithms as well as lower bounds. The large variety of tools and techniques used, the ease with which tools from one area are applied in another, and the overall quality of presentation of the results indicate the high level of general theoretical knowledge of the candidate in the discipline and their ability to do high-quality scientific work. The novelty of the approach, linking the parameterized complexity of a problem with an asymptotic graph parameter, is also commendable. I would like to emphasize that I had reviewed over a dozen doctoral dissertations from various universities, and none of them impressed me as much as this one.

Hence, based on my evaluation of the dissertation, I recommend it to be unconditionally **accepted for the final defense**. Furthermore, for all the reasons outlined above, I suggest the dissertation to be **distinguished**.

Sincerely,

Prof. Martin Milanič, PhD

Martin Milanič