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Reviewer's report of the PhD dissertation *„Graph Homomorphisms – Exploring the Boundaries of Tractability"* by Marta Piecyk

The doctoral dissertation of mgr. Marta Piecyk concerns the computational complexity of the graph homomorphism problem. This is a well-known generalization of the graph coloring problem, where the task is to determine whether there exists a homomorphism from a graph G to a graph H, that is, a function $h : V(G) \to V(H)$ that maps the endpoints of every edge of G to the endpoints of an edge of H. In the dissertation, the graph H is assumed to be fixed, so the focus is on the complexity with respect to the number of vertices n of G. According to a classical result by Hell and Nešetřil, the problem is NP-hard unless H is bipartite. Therefore, the dissertation focuses on finding new parameterized (FPT) and polynomial-time algorithms under additional assumptions. On the other hand, when such algorithms seem impossible, the author establishes lower bounds. Overall, this topic fits within the mainstream of contemporary computer science, as evidenced by the strong venues of publication (conferences such as ICALP, ESA, MFCS).

Overview of the results

The first group of results (Chapter 4) concerns parameterized complexity with respect to the cutwidth t. In the special case where H is a clique, an algorithm with running time $c^t n^{O(1)}$ for some constant c independent of the size of the clique is known. The author showed in her master's thesis that, assuming the Exponential Time Hypothesis (ETH), such an algorithm cannot be achieved for the general homomorphism problem. However, if one allows each graph H to have its own constant c_H, it is possible to ob-

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tain a running time of $c_H^t n^{O(1)}$; a classical dynamic programming algorithm achieves $c_H = |V(H)|$. As can be seen in the case of cliques, this is not always the optimal constant. The goal of the first part of the dissertation, based on a publication at the prestigious ICALP conference, is to find the optimal constant c_H for each graph H. This goal is only partially achieved in the dissertation. Namely, the author introduces a convincing candidate for the constant c_H: a value called $\mathrm{mimsup}(H)$, which can be expressed as the supremum of the size of the maximum induced matching in an appropriate family of powers of the graph H, or equivalently, in algebraic terms. The main reasons for mimsup being the right answer is twofold. The first reason is a DP algorithm based on the concept of representative sets. It turns out that mimsup captures well the minimum amount of information (called a representative set) that is sufficient for the DP to work. Unfortunately, finding such a set efficiently is still an open problem. The author shows a partial success in attacking this problem: by introducing a proxy parameter $\mathrm{him}(H) \leq \mathrm{mimsup}(H)$ an algorithm with running time $2^{O(t \log t\, \mathrm{him}(H))} n^{O(1)}$ is shown. Here, the exponent is $O(\log t\, \mathrm{him}(H)/\log \mathrm{him}(H))$ larger than desired, though in some scenarios it is an improvement over the naive DP. The second argument for c_H being mimsup is more concrete: the author shows conditional lower bounds based on ETH and SETH, in particular the latter implies that $c_H \geq \mathrm{mimsup}$ for so-called projective cores, which is a natural class of almost all graphs. Chapter 4 contains also a few minor results obtained in the hunt for a better understanding of the mimsup parameter and different attempts to make the DP work fast.

The second group of results (Chapter 5) is inspired by the classical open problem of determining if 3-coloring diameter 2 graphs is NP-complete. While the problem is still open, in 2013 Mertzios and Spirakis showed a subexponential $2^{O(\sqrt{n}\log n)}$ algorithm, proved that the diameter 3 case is NP-complete and diameter 4 graphs do not admit subexponential algorithms. The dissertation improves the diameter 2 case to $2^{O(n^{1/3}\log^2 n)}$ and provides the first subexponential algorithm for diameter 3, in time $2^{O(n^{2/3}\log^{2/3} n)}$. This is achieved by a number of clever combinatorial observations and interesting techniques including a probabilistic argument for existence of a set of desired properties and a non-standard instance measure.

The rest of Chapter 5 explores another question inspired by the open pro-

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blem of 3-coloring diameter 2 graphs. What if we look at it as a homomorphism problem and consider different target graphs than a triangle? First, it is shown that the problem is P-time for triangle-free graphs. Next, the author considers odd cycles. For each odd cycle C_{2k+1}, $k \geq 2$, the author shows a diameter upper bound up to which the problem is P-time, and a lower bound which excludes P-time algorithm or even subexponential algorithms, assuming ETH. These bounds leave a gap of size $k + 1$, and for a single cycle length in the gap the author shows a subexponential algorithm, which generalizes the ideas from the algorithms from the first part of the chapter. It is worth mentioning that the contents of this part originates from a single-author paper.

Evaluation

With no doubts the results of the PhD thesis of Marta Piecyk constitute an important and impressive contribution to computer science. The problems studied are well-motivated. It should be observed that in order to extend our understanding of big open questions, the thesis creates a handful of new auxiliary problems, which are interesting themselves and most likely will be studied by other researchers. Another striking aspect of the thesis is the wide variety of methods applied. Indeed, creating this thesis required knowledge of advanced dynamic programming (representative sets), applications of algebra in graph theory (Kronecker products, matrix ranks), probabilistic method, measure-and-conquer branching algorithms, the art of crafting delicate ETH- and SETH-based reductions, and many more. Lastly, the thesis is well-written (modulo a few typos).

To sum up, the thesis shows that the author is a versatile researcher with impressive technical skills. In my opinion, the dissertation **fulfills** the requirements for PhD theses in mathematical or computer sciences formulated by law and accepted in the community. Moreover, I recommend to distinguish the thesis (pol. wyróżnienie).

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